Line Integrals

Definition Let E be a subset in $\mathbb{R}^m$. A vector field on E is a vector (or vector-valued) function $F : E \to \mathbb{R}^n$ defined by

$$F(x) = (F_1(x), F_2(x), \ldots, F_n(x)) \in \mathbb{R}^n \quad \text{for each } x = (x_1, x_2, \ldots, x_m) \in E.$$

Definition Let C be a plane curve defined by the parametric equations $r(t) = \big(x(t)\,,\, y(t)\big)$, $t \in [a, b]$. Then

- C is called a smooth curve if $r'(t)$ is continuous and $r'(t) \neq 0$ for all $t \in [a, b]$.
- C is called a piecewise smooth curve if there exists a partition

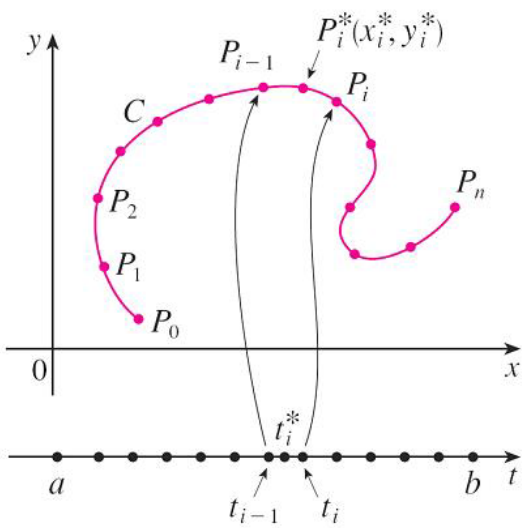
  $$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

  such that $r'(t) \neq 0$ is continuous for all $t \in (t_{i-1}, t_i)$ and $\lim\limits_{t \to t_i^{\pm}} r'(t)$ exists for each $1 \leq i \leq n$.

Definition Let C be a smooth plane curve given by the vector function $r(t) = \big(x(t)\,,\, y(t)\big)$, $t \in [a, b]$, let f be a function defined on C and let $s(t) = \displaystyle\int_a^t |r'(u)|\, du$. Then the line integral of f along C is defined by

$$\int_C f(x, y)\, ds = \int_a^b f(r(t))\, |r'(t)|\, dt = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i \quad \text{if this limit exists,}$$

where $ds = |r'(t)|\, dt = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt$, $P_i^*(x_i^*, y_i^*) = r(t_i^*) \in \widehat{P_{i-1}P_i}$ and Δs_i is the length of the subarc $\widehat{P_{i-1}P_i}$ from $P_{i-1} = r(t_{i-1})$ to $P_i = r(t_i)$.



In general, if C is a (piecewise) smooth curve in $\mathbb{R}^m$ given by the vector function $r(t) = (x_1(t), \ldots, x_m(t))$, $t \in [a, b]$ and

- if f is a function (scalar field) defined on C, then the line integral of f along C is defined by

$$\begin{aligned}\int_C f(x_1, x_2, \dots, x_m)\, ds &= \int_a^b f(r(t))\, |r'(t)|\, dt \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{1i}^*, x_{2i}^*, \dots, x_{mi}^*)\, \Delta s_i \quad \text{if this limit exists.}\end{aligned}$$

- if $F = (F_1, \dots, F_m)$ is a continuous vector field defined on C, then the line integral of F along C is defined by

$$\int_C F(r) \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt,$$

  where $F(r(t)) \cdot r'(t)$ denotes the inner product of $F(r(t))$, $r'(t) \in \mathbb{R}^m$.

Remarks Let C (piecewise) smooth curve defined by $r(t)$, $t \in [a, b]$.

(a) If $\phi : [c, d] \to [a, b]$ is a continuously differentiable, orientation preserving onto map such that $\phi'(u) > 0$, $\phi(c) = a$, $\phi(d) = b$, then C is given by the vector function $R(u)$ defined by

$$C\ :\ R(u) = r(\phi(u)), \qquad u \in [c, d],$$

and since

$$\begin{aligned}\int_C F(R) \cdot dR &= \int_c^d F(R(u)) \cdot R'(u)\, du \\ &= \int_c^d [F(r(\phi(u))) \cdot r'(\phi(u))]\ \phi'(u)\, du \\ &\quad \text{Set } t = \phi(u) \implies dt = \phi'(u)\, du,\ \phi(c) = a,\ \phi(d) = b \\ &= \int_a^b \big[F(r(t)) \cdot r'(t)\big]\, dt \ = \int_C F(r) \cdot dr\end{aligned}$$

the line integral is left invariant by every orientation-preserving change of parameter.

(b) For $u \in [a, b]$, let $\phi(u) = a + b - u$ and let $-C$ be a curve defined by

$$-C\ :\quad R(u) = r(\phi(u)) = r(a + b - u) \quad \text{for } u \in [a, b].$$

Since $\phi : [a, b] \to [a, b]$ is onto, $\phi'(u) = -1$, $\phi(a) = b$ and $\phi(b) = a$, the curve $-C$ denotes the same curve traversed in the opposite direction and

$$\begin{aligned}\int_{-C} F(R) \cdot dR &= \int_a^b F(R(u)) \cdot R'(u)\, du \\ &= \int_a^b \big[F(r(\phi(u))) \cdot r'(\phi(u))\big]\ \phi'(u)\, du \\ &\quad \text{Set } t = \phi(u) \implies dt = \phi'(u)\, du,\ \phi(a) = b,\ \phi(b) = a \\ &= \int_b^a \big[F(r(t)) \cdot r'(t)\big]\, dt = -\int_a^b \big[F(r(t)) \cdot r'(t)\big]\, dt \ = -\int_C F(r) \cdot dr\end{aligned}$$

Examples

1. Evaluate $\int_C (2+x^2y)\,ds$, where C is the upper half of the unit circle $x^2+y^2=1$.
2. The center of mass of the wire C with density function $\rho(x,y)$, $(x,y)\in C$, is located at the point

$$\bar{x}=\frac{1}{m}\int_C x\rho(x,y)\,ds \quad \bar{y}=\frac{1}{m}\int_C y\rho(x,y)\,ds, \quad \text{where} \quad m=\int_C \rho(x,y)\,ds = \text{ total mass of } C.$$

3. Evaluate $\int_{C_1} y^2\,dx + x\,dy$, where C_1 is the line segment from $(-5,-3)$ to $(0,2)$, and evaluate $\int_{C_2} y^2\,dx + x\,dy$, where C_2 is the arc of the parabola $x=4-y^2$ from $(-5,-3)$ to $(0,2)$.
4. Evaluate $\int_C y\sin z\,ds$, where C is the circular helix given by the equations $x=\cos t$, $y=\sin t$, $z=t$, $0\le t\le 2\pi$.
5. Find the work done $\int_C F(r)\cdot dr$ by the force field $F(x,y)=x^2\,\mathbf{i}-xy\,\mathbf{j}$ in moving a particle along the quarter-circle $C:\ r(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}$, $0\le t\le \pi/2$.

Definition Let E be a subset of $\mathbb{R}^n$ and let $F=(F_1,F_2,\ldots,F_n): E\to\mathbb{R}^n$ be a vector field defined on E. Then F is called a conservative vector field if there exists a function $f:\mathbb{R}^n\to\mathbb{R}$ such that

$$F(x)=\nabla f(x) \quad \text{for all } x\in E.$$

Note that if F is a continuously differentiable conservative vector field, then f has continuous 2$^{\text{nd}}$ order partial derivatives $f_{x_ix_j}=f_{x_jx_i}$ and

$$\frac{\partial F_i}{\partial x_j}=f_{x_ix_j}=f_{x_jx_i}=\frac{\partial F_j}{\partial x_i} \implies \frac{\partial F_i}{\partial x_j}=\frac{\partial F_j}{\partial x_i} \quad \text{for each } 1\le i,\ j\le n \text{ (integrability conditions)}$$

In this situation, the function f is called a potential function for F.

Definition Let F be a continuous vector field defined on D, and let C_1 and C_2 be paths in D having the same initial points and the same terminal points. Then the line integral $\int_C F\cdot dr$ is called independent of path if

$$\int_{C_1} F\cdot dr = \int_{C_2} F\cdot dr$$

Fundamental Theorem for Line Integrals Let U be an open subset of $\mathbb{R}^m$, let $f:U\to\mathbb{R}$ be a continuously differentiable scalar field (function) and let $C: r=r(u)$, $u\in[a,b]$ be a (piecewise) smooth curve that begins at $p=r(a)$ and ends at $q=r(b)$. Then the line integral of (a conservative field) ∇f along C satisfies that

$$\int_C \nabla f(r)\cdot dr = f(r(b))-f(r(a))=f(q)-f(p)$$

and is independent of the choice of paths in U joining from p to q.

Proof

Case 1: If C is smooth, then

$$\int_C \nabla f(r)\cdot dr = \int_a^b \nabla f(r(u))\cdot r'(u)\,du = \int_a^b \left[\frac{d}{du} f(r(u))\right] du = f(r(b)) - f(r(a)).$$

Case 2: If C a (piecewise) smooth and $\{a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b\}$ is a partition of $[a, b]$ such that

$$C = \bigcup_{i=1}^{n} C_i \quad \text{where} \quad C_i = \{r(t) \mid t \in [a_{i-1}, a_i]\} \text{ is smooth for } 1 \le i \le n,$$

then

$$\begin{aligned}\int_C \nabla f(r)\cdot dr &= \int_{\bigcup_{i=1}^n C_i} \nabla f(r)\cdot dr = \sum_{i=1}^{n} \int_{C_i} \nabla f(r)\cdot dr = \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} \nabla f(r(u))\cdot r'(u)\,du \\ &\overset{\text{Case 1}}{=} \sum_{i=1}^{n} f(r(a_i)) - f(r(a_{i-1})) = f(r(b)) - f(r(a)).\end{aligned}$$

Remarks

(a) This is a generalization of the Fundamental Theorem of Calculus since if $g : [a, b] \to \mathbb{R}$ is a continuous function with an anti-derivative $G(x)$ on (a, b) and if C is the line segment given by $r(x) = x,\ x \in [a, b] \subset \mathbb{R}$, then

$$G(b) - G(a) = \int_C \nabla G(r)\cdot dr = \int_a^b \frac{d}{dx} G(x)\,dx = \int_a^b g(x)\,dx$$

(b) If $C : r = r(u),\ u \in [a, b]$ is a (piecewise) smooth closed curve and if f is continuously differentiable on an open set U containing C, then

$$\int_C \nabla f(r)\cdot dr = f(r(b)) - f(r(a)) = 0 \quad \text{since } r(b) = r(a).$$

This implies that if $F = \nabla f$ is conservative and continuous in U, the line integral $\int_C F(r)\cdot dr = 0$ along any (piecewise) smooth closed curve C in U.

Examples

1. Find the work done by the gravitational field $F(X) = -\frac{mMG}{|X|^3}X,\ X = (x, y, z) \in \mathbb{R}^3$, in moving a particle with mass m from the point $(3, 4, 12)$ to the point $(2, 2, 0)$ along a piecewise-smooth curve C.
2. Determine whether or not the given vector field is conservative.
   - $F(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$.
   - $F(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$.

Definition A continuous curve $C : r = r(u)\ u \in [a, b]$ is called simple if

$$r(u) \ne r(t) \quad \text{for all } a \le t < u < b \iff \text{for all } u \ne t \in [a, b).$$

A Jordan curve is a plane curve that is both closed and simple.

Definition A region D is called (path) connected if any two points in D can be joined by a path that lies in D, i.e. D is (path) connected if for all $p, q \in D$, there exists a continuous map $r : [a, b] \to D$ from $[a, b]$ into D such that $r(a) = p$ and $r(b) = q$.

Definition Let D be a region in the plane. Then D is called simply-connected if every simple closed curve in D encloses only points that are in D.

Remark If $F = \nabla f$ is conservative and continuous in D, and if C_1 and C_2 are paths in D having the same initial points and the same terminal points. Then $C_1 \cup (-C_2)$ is a closed curve in D, and

$$\begin{aligned} 0 &= \int_{C_1 \cup (-C_2)} F \cdot dr = \int_{C_1} F \cdot dr + \int_{-C_2} F \cdot dr = \int_{C_1} F \cdot dr - \int_{C_2} F \cdot dr \\ \Longrightarrow & \int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr \text{ (independent of path)} \end{aligned}$$

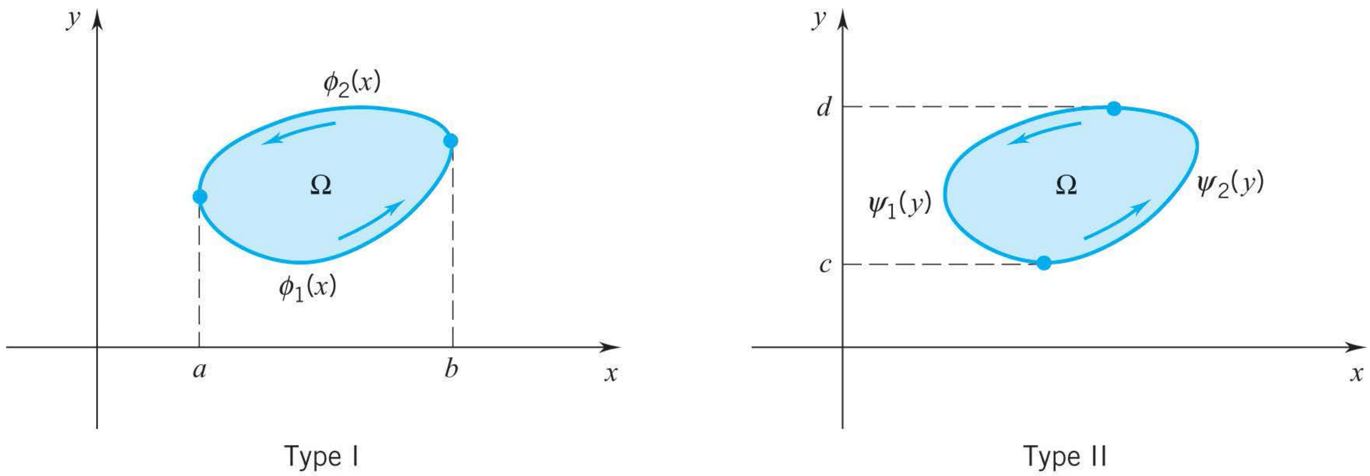
Green's Theorem Let $F = (P, Q)$ be a vector field on an simply-connected region Ω. Suppose that P and Q have continuous first-order partial derivatives on an open set that contains Ω, then

$$\iint_\Omega \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\, dy = \oint_C P dx + Q dy,$$

where the boundary $C = \partial\Omega$ is oriented in the positive direction such that Ω is on the left-hand-side when traversing along C, and $\oint_C$ denotes the line integral along C in the positively orientation.

Proof Suppose that Ω is an elementary region given by

$$\{(x, y) \mid a \le x \le b,\ \phi_1(x) \le y \le \phi_2(x)\} \text{ or } \{(x, y) \mid \psi_1(y) \le x \le \psi_2(y),\ c \le x \le d\},$$



Ω is an elementary region: it is both of Type I and Type II

such that the boundary $\partial\Omega = C = C_1 \cup C_2 = C_3 \cup C_4$, where

$$\begin{aligned} C_1 &= \{(x, y) \mid y = \phi_1(x),\ a \le x \le b\}, \quad C_2 = \{(x, y) \mid y = \phi_2(x),\ a \le x \le b\}, \\ C_3 &= \{(x, y) \mid x = \psi_1(y),\ c \le y \le d\}, \quad C_4 = \{(x, y) \mid x = \psi_2(y),\ c \le y \le d\}. \end{aligned}$$

Since

$$\begin{aligned}\iint_\Omega \frac{\partial Q}{\partial x}\,dx\,dy &= \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d Q(\psi_2(y),y)\,dy - \int_c^d Q(\psi_1(y),y)\,dy \\ \iint_\Omega -\frac{\partial P}{\partial y}\,dy\,dx &= -\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y}\,dy\,dx = \int_a^b P(x,\phi_1(x))\,dx - \int_a^b P(x,\phi_2(x))\,dx\end{aligned}$$

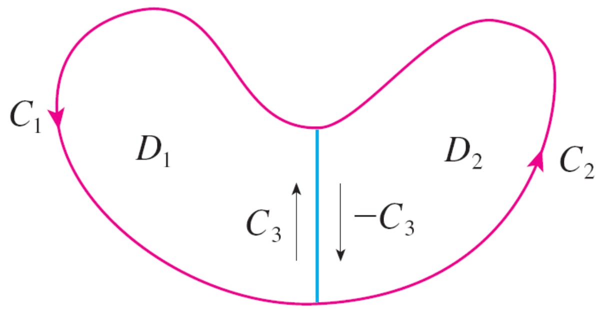
and

$$\begin{aligned}\oint_C Q\,dy &= \int_{C_4\cup(-C_3)} Q\,dy = \int_{C_4} Q\,dy - \int_{C_3} Q\,dy = \int_c^d Q(\psi_2(y),y)\,dy - \int_c^d Q(\psi_1(y),y)\,dy \\ \oint_C P\,dx &= \int_{C_1\cup(-C_2)} P\,dx = \int_{C_1} P\,dx - \int_{C_2} P\,dx = \int_a^b P(x,\phi_1(x))\,dx - \int_a^b P(x,\phi_2(x))\,dx\end{aligned}$$

we have

$$\iint_\Omega \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy = \oint_C Pdx + Qdy.$$

Remark Suppose that D_1 and D_2 are simply-connected regions with boundaries $\partial D_1 = C_3 \cup C_1$ and $\partial D_2 = C_2 \cup (-C_3)$ respectively and suppose that $D = D_1 \cup D_2$ has the boundary $C = \partial D = C_1 \cup C_2$.



Since

$$\iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx = \iint_{D_1\cup D_2} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx = \iint_{D_1} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx + \iint_{D_2} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx$$

$$\oint_C P\,dx + Q\,dy = \oint_{C_1\cup C_2\cup C_3\cup(-C_3)} P\,dx + Q\,dy = \oint_{C_1\cup C_3} P\,dx + Q\,dy + \oint_{C_2\cup(-C_3)} P\,dx + Q\,dy$$

and since D_1, D_2 are simple regions as in the preceding proof, we have

$$\begin{aligned}\iint_{D_1} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx &= \oint_{C_1\cup C_3} Pdx + Qdy \\ \iint_{D_2} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx &= \oint_{C_2\cup(-C_3)} Pdx + Qdy\end{aligned}$$

This proves the Green's Theorem on a more general region D

$$\iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy = \oint_{C_1\cup C_2} Pdx + Qdy = \oint_{\partial D} Pdx + Qdy.$$

Theorem Let $F = (P, Q)$ be a vector field on a simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D \quad \text{(integrability condition)}$$

Then F is conservative, i.e. there is a continuously differentiable function $f : D \to \mathbb{R}$ such that $F = \nabla f$ on D.

Proof Fix a point $p = (a, b) \in D$. For any $(s, t) \in D$, let $f : D \to \mathbb{R}$ be defined by

$$f(s,t) = \int_C P\,dx + Q\,dy, \quad \text{where } C \text{ is a piecewise smooth path in } D \text{ from } p \text{ to } (s,t).$$

This is well defined since if C_1 is another piecewise smooth path in D from p to (s, t) such that $C \cup (-C_1)$ is a simple closed curve enclosing a region $E \subset D$. By the Green's Theorem, we have

$$\int_C P\,dx + Q\,dy - \int_{C_1} P\,dx + Q\,dy = \oint_{C\cup(-C_1)} P\,dx + Q\,dy = \iint_E \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy = 0,$$

and

$$\int_C P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy \quad \text{(independent of path)}$$

Hence we may simply rewrite the definition of f at each point $(s, t) \in D$ as follows

$$f(s,t) = \int_p^{(s,t)} P\,dx + Q\,dy,$$

where the line integral is integrated along any piecewise smooth path in D from p to (s, t). For each $(s, t) \in D$ and for any sufficiently small h, k such that $(s + h, t + k) \in D$, since

$$\begin{aligned}
f(s+h,t) &= \int_p^{(s+h,t)} P\,dx + Q\,dy \\
&= \int_p^{(s,t)} P\,dx + Q\,dy + \int_{(s,t)}^{(s+h,t)} P\,dx + Q\,dy \\
&\quad \text{since } y = t \text{ (const.) on } (s,t) \to (s+h,t) \implies dy = 0 \\
&= f(s,t) + \int_{(s,t)}^{(s+h,t)} P\,dx \\
&= f(s,t) + \int_s^{s+h} P(x,t)\,dx,
\end{aligned}$$

$$\begin{aligned}
f(s,t+k) &= \int_p^{(s,t+k)} P\,dx + Q\,dy \\
&= \int_p^{(s,t)} P\,dx + Q\,dy + \int_{(s,t)}^{(s,t+k)} P\,dx + Q\,dy \\
&\quad \text{since } x = s \text{ (const.) on } (s,t) \to (s,t+k) \implies dx = 0 \\
&= f(s,t) + \int_{(s,t)}^{(s,t+k)} Q\,dy \\
&= f(s,t) + \int_t^{t+k} Q(s,y)\,dy,
\end{aligned}$$

by the Fundamental Theorem of Calculus, we have

$$\begin{aligned} f_x(s,t) &= \lim_{h\to 0} \frac{f(s+h,t)-f(s,t)}{h} = \lim_{h\to 0} \frac{\int_s^{s+h} P(x,t)\,dx}{h} = P(s,t) \\ f_y(s,t) &= \lim_{k\to 0} \frac{f(s,t+k)-f(s,t)}{k} = \lim_{k\to 0} \frac{\int_t^{t+k} Q(s,y)dy}{k} = Q(s,t) \end{aligned}$$

Hence

$$F = (P,Q) = (f_x, f_y) = \nabla f \quad \text{on } D \implies F \text{ is conservative on } D.$$

Examples

1. Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$ and from $(0,1)$ to $(0,0)$.
   Solution: By the Green's Theorem $\int_C x^4\,dx + xy\,dy = \int_0^1 \int_0^{1-x} (y-0)\,dy\,dx$.
2. Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. By the Green's Theorem, the area of D is given by

$$A(D) = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx,$$

   where C is positively oriented (i.e. move along C in the direction so that D is on the left).
   Use the formula to find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
   Solution: By the Green's Theorem, $A = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} a\,b\,d\theta$
3. Evaluate $\int_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2+y^2=1$ and $x^2+y^2=4$.
   Solution: By the Green's Theorem $\int_C y^2\,dx + 3xy\,dy = \iint_D y\,dA = \int_0^{\pi}\int_1^2 r^2 \sin\theta\,dr\,d\theta$

Curl and Divergence

Definition Let E be a subset of $\mathbb{R}^3$ and let $F = (F_1, F_2, F_3)$ be a vector field, f be a differentiable function defined on E. Let the vector differential operator ∇ ("del") be defined by

$$\nabla = \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2} + \mathbf{k}\frac{\partial}{\partial x_3}$$

Suppose that the partial derivatives of F_1, F_2, F_3, and f all exist, then the curl of F is the vector field on $\mathbb{R}^3$ defined by

$$\begin{aligned}\operatorname{curl} F &= \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)\mathbf{k} \\ &= \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)\end{aligned}$$

the gradient of f is a vector field on $\mathbb{R}^3$ defined by

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x_1}\mathbf{i} + \frac{\partial f}{\partial x_2}\mathbf{j} + \frac{\partial f}{\partial x_3}\mathbf{k} = \left(f_{x_1}, f_{x_2}, f_{x_3}\right)$$

and the divergence of F is a function on $\mathbb{R}^3$ defined by

$$\operatorname{div} F = \nabla \cdot F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}, \quad \text{where } \nabla = \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2} + \mathbf{k}\frac{\partial}{\partial x_3}$$

Example Let $F(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$. (1) Find $\operatorname{curl} F$. (2) Find $\operatorname{div} F$.

Theorem Let $F = (F_1, F_2, F_3)$ be a vector field defined on $\mathbb{R}^3$. Suppose that the component functions have continuous partial derivatives. Then $\operatorname{curl} F = 0$ if and only if F is a conservative vector field.

Proof If

$$\operatorname{curl} F = \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) = (0, 0, 0) \quad \text{on } \mathbb{R}^3,$$

the integrability conditions hold and there is a continuously differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $F = \nabla f$ on $\mathbb{R}^3$, i.e. F is a conservative vector field.

Conversely, if F is a conservative vector field such that there is a continuously differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$ having has continuous 2^{nd} order partial derivatives on $\mathbb{R}^3$, then

$$\operatorname{curl} F = \operatorname{curl} \nabla f = 0 \quad \text{on } \mathbb{R}^3.$$

Theorem Let $F = (F_1, F_2, F_3)$ be a vector field defined on $\mathbb{R}^3$. Suppose that the component functions have continuous 2^{nd} order partial derivatives. Then

$$\operatorname{div} \operatorname{curl} F = 0 \quad \text{on } \mathbb{R}^3.$$

Divergence Theorem Let $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid a \le x_1 \le b,\ c \le x_2 \le d,\ e \le x_3 \le f\}$ be a closed cell in $\mathbb{R}^3$ and let $F = (F_1, F_2, F_3)$ be a continuous vector field on W. Suppose that $\dfrac{\partial F_1}{\partial x_1}$, $\dfrac{\partial F_2}{\partial x_2}$ and $\dfrac{\partial F_3}{\partial x_3}$ are continuous on an open set U containing W. Then

$$\iiint_W \operatorname{div} F\, dV = \iint_{\partial W} F \cdot n\, dA,$$

where $n = n(p)$ denotes the unit outward normal vector to ∂W at $p \in \partial W$.

Proof Since the boundary ∂W of W consists of 6 faces $S_1 \cup S_2 \cup \cdots \cup S_6$, where

$$\begin{aligned}
S_1 &= \{(x_1,x_2,x_3) \in W \mid x_1 = a\} \Longrightarrow \text{if } p \in S_1 \text{ then } n(p) = (-1,0,0)\\
S_2 &= \{(x_1,x_2,x_3) \in W \mid x_1 = b\} \Longrightarrow \text{if } p \in S_2 \text{ then } n(p) = (1,0,0)\\
S_3 &= \{(x_1,x_2,x_3) \in W \mid x_2 = c\} \Longrightarrow \text{if } p \in S_3 \text{ then } n(p) = (0,-1,0)\\
S_4 &= \{(x_1,x_2,x_3) \in W \mid x_2 = d\} \Longrightarrow \text{if } p \in S_4 \text{ then } n(p) = (0,1,0)\\
S_5 &= \{(x_1,x_2,x_3) \in W \mid x_3 = e\} \Longrightarrow \text{if } p \in S_5 \text{ then } n(p) = (0,0,-1)\\
S_6 &= \{(x_1,x_2,x_3) \in W \mid x_3 = f\} \Longrightarrow \text{if } p \in S_6 \text{ then } n(p) = (0,0,1)
\end{aligned}$$

we have

$$\begin{aligned}
&\iiint_W \operatorname{div} F\, dV = \iiint_W \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}\right) dV\\
&= \int_e^f \int_c^d \int_a^b \frac{\partial F_1}{\partial x_1}\, dx_1\, dx_2\, dx_3 + \int_e^f \int_a^b \int_c^d \frac{\partial F_2}{\partial x_2}\, dx_2\, dx_1\, dx_3 + \int_a^b \int_c^d \int_e^f \frac{\partial F_3}{\partial x_3}\, dx_3\, dx_2\, dx_1\\
&= \int_e^f \int_c^d (F_1(b,x_2,x_3) - F_1(a,x_2,x_3))\, dx_2\, dx_3 + \int_e^f \int_a^b (F_2(x_1,d,x_3) - F_2(x_1,c,x_3))\, dx_1\, dx_3\\
&+ \int_a^b \int_c^d (F_3(x_1,x_2,f) - F_3(x_1,x_2,e))\, dx_2\, dx_1\\
&= \iint_{S_2} F_1(b,x_2,x_3)\, dx_2\, dx_3 - \iint_{S_1} F_1(a,x_2,x_3)\, dx_2\, dx_3 + \iint_{S_4} F_2(x_1,d,x_3)\, dx_1\, dx_3\\
&- \iint_{S_3} F_2(x_1,c,x_3)\, dx_1\, dx_3 + \iint_{S_6} F_3(x_1,x_2,f)\, dx_2\, dx_1 - \iint_{S_5} F_3(x_1,x_2,e))\, dx_2\, dx_1\\
&= \iint_{S_2} F \cdot (1,0,0)\, dx_2\, dx_3 + \iint_{S_1} F \cdot (-1,0,0)\, dx_2\, dx_3 + \iint_{S_4} F \cdot (0,1,0)\, dx_1\, dx_3\\
&+ \iint_{S_3} F \cdot (0,-1,0)\, dx_1\, dx_3 + \iint_{S_6} F \cdot (0,0,1)\, dx_2\, dx_1 + \iint_{S_5} F \cdot (0,0,-1)\, dx_2\, dx_1\\
&= \iint_{S_2 \cup S_1 \cup S_4 \cup S_3 \cup S_6 \cup S_5} F \cdot n\, dA = \iint_{\partial W} F \cdot n\, dA
\end{aligned}$$

Remark In general, if R is a regular region in $\mathbb{R}^n$ with piecewise smooth boundary ∂R, and if $F = (F_1, \ldots, F_n)$ is a continuously differentiable vector field on $R \cup \partial R$, then

$$\int_R \operatorname{div} F\, dV = \int_{\partial R} F \cdot \nu\, dS$$

where $\nu = \nu(x)$ denotes the unit outward normal vector to ∂R at $x \in \partial R$ and dS_x denotes the volume element of ∂R at $x \in \partial R$.

Corollary (Green's Theorem) Let R be a regular region in $\mathbb{R}^2 = x_1x_2$-plane with piecewise smooth boundary ∂R, and let $F = (F_1, F_2) : R \cup \partial R \to \mathbb{R}^2$ be a continuously differentiable vector field on $R \cup \partial R$. Then

$$\int_{\partial R} F \cdot dx = \iint_R \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dA, \quad \text{where } dx = (dx_1, dx_2)$$

Proof Note that if $r(t) = (x_1(t), x_2(t)) : [a, b] \to \partial R$ is a parametrization (or coordinate functions) of ∂R, then

$$\int_{\partial R} F \cdot dx = \int_{r([a,b])} (F_1, F_2) \cdot (dx_1, dx_2) = \int_a^b (F_1, F_2) \cdot \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}\right) dt$$

Since $r'(t) = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}\right)$ is tangent to ∂R at $p = r(t)$, the vector $\nu(p) = \left(\frac{dx_2}{dt}, -\frac{dx_1}{dt}\right)$ is a normal vector there.

Let $G = (G_1, G_2) = (F_2, -F_1)$. Then G is a continuously differentiable vector field on $R \cup \partial R$, and

$$\int_{\partial R} F \cdot dx = \int_{\partial R} (F_1, F_2) \cdot (dx_1, dx_2) = \int_{\partial R} (F_2, -F_1) \cdot (dx_2, -dx_1) = \int_a^b (G_1, G_2) \cdot \nu \, dt,$$

and, by the divergence theorem, we have

$$\int_a^b (G_1, G_2) \cdot \nu \, dt = \int_{\partial R} G \cdot \nu = \iint_R \operatorname{div} G \, dA = \iint_R \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dA$$

Definition If F is a continuous vector field defined on an oriented surface S with unit normal vector n, then the surface integral of F over S is

$$\iint_S F \cdot d\mathbf{S} = \iint_S F \cdot n \, dS \quad = \text{the flux of } F \text{ across } S,$$

where $d\mathbf{S}$ is the vector area element of S, dS is the area element of S, $n = n(p)$ is the unit outward normal vector to S at p. This integral is also called the flux of F across S.

Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let F be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C F \cdot dr = \iint_S \operatorname{curl} F \cdot d\mathbf{S} \quad = \iint_R (\nabla \times F) \cdot \frac{\partial r}{\partial s_1} \times \frac{\partial r}{\partial s_2} \, dA,$$

where

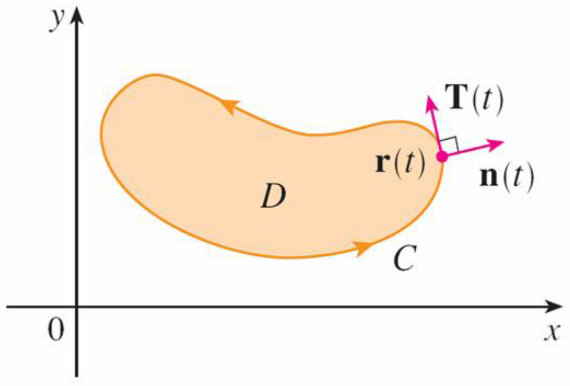
- $r(s_1, s_2) = (x_1(s_1, s_2), x_2(s_1, s_2), x_3(s_1, s_2)) : R \to S$ is a smooth parametrization that maps a simple, closed, piecewise-smooth bounded region R, in s_1s_2-plane, to a surface S in $x_1x_2x_3$-space and $r : \partial R \to C$ maps the boundary ∂R of R onto C,
- $dr = \frac{\partial r}{\partial s_1} ds_1 + \frac{\partial r}{\partial s_2} ds_2$ is the tangent vector length element of C,
- $d\mathbf{S}$ is the vector area element of S and dA is the area element of R.

Remark If C is a smooth simple closed curve given by the vector equation

$$r(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} = \big(x(t), y(t), 0\big) \quad \text{for } a \le t \le b$$

then

$$T(t) = \frac{x'(t)}{|r'(t)|}\,\mathbf{i} + \frac{y'(t)}{|r'(t)|}\,\mathbf{j} \quad \text{and} \quad n(t) = \frac{y'(t)}{|r'(t)|}\,\mathbf{i} - \frac{x'(t)}{|r'(t)|}\,\mathbf{j}.$$



are respectively the unit tangent vector and the outward unit normal vector to C at $r(t)$, and

$$n(t)\times T(t) = \frac{(x'(t))^2+(y'(t))^2}{|r'(t)|^2}(\mathbf{i}\times\mathbf{j}) = \mathbf{i}\times\mathbf{j}=\mathbf{k} \quad \text{for each } a\le t\le b$$
$$\Longrightarrow \quad D\subset xy\text{-plane} \implies dz=0 \text{ on } D,$$

Since

$$\operatorname{curl} F = \left(\frac{\partial F_3}{\partial x_2}-\frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3}-\frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1}-\frac{\partial F_1}{\partial x_2}\right)$$

we have

$$\begin{aligned}
\iint_D \operatorname{curl} F &= \mathbf{i}\iint_D\left[\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right]dy\,dz+\mathbf{j}\iint_D\left[\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right]dx\,dz+\mathbf{k}\iint_D\left[\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right]dx\,dy\\
&= \left(\iint_D\left[\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right]dx\,dy\right)\mathbf{k} \quad \text{since } dz=0 \text{ on } D\\
&= \left(\oint_C F_1dx+F_2dy\right)\mathbf{k} \quad \text{by the Green's Theorem}\\
&= \left(\oint_C F(r)\cdot dr\right) n(t)\times T(t)
\end{aligned}$$

perpendicular to the plane spanned by $T,\ n$

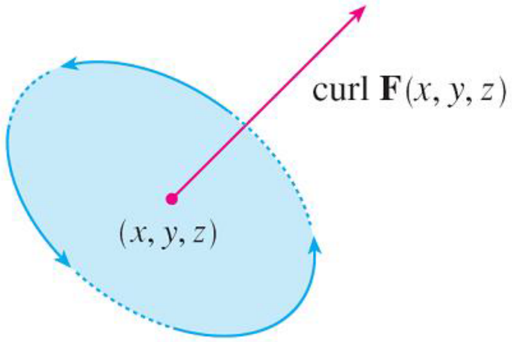
so, by setting $N=T\times n$ along C, we obtain a positively oriented basis $\{N,\,n,\,T\}$ for $\mathbb{R}^3$ and note that the curl F at a point $p=(x,y,z)$ can be defined by

$$\operatorname{curl} F(p) = \lim_{A\to 0}\frac{1}{A}\iint_D \operatorname{curl} F = \lim_{A\to 0}\left(\frac{1}{A}\oint_C F(r)\cdot dr\right) N \ \perp \ \text{the plane containing } C$$

where A is the area of D and $\oint_C F(r)\cdot dr$, a line integral along the boundary of D, measures the velocity of particles move around the axis.

Proof of Stokes' Theorem Since $C=r(\partial R)$ and $dr=\dfrac{\partial r}{\partial s_1}ds_1+\dfrac{\partial r}{\partial s_2}ds_2$, we have

$$\int_C F\cdot dr = \int_{\partial R}\left(F\cdot\frac{\partial r}{\partial s_1}ds_1+F\cdot\frac{\partial r}{\partial s_2}ds_2\right) \quad \text{by the definition of line integral.}$$



Setting

$$G_1 = F \cdot \frac{\partial r}{\partial s_1} \quad \text{and} \quad G_2 = F \cdot \frac{\partial r}{\partial s_2},$$

and by the Green's Theorem, we have

$$\int_C F \cdot dr = \int_{\partial R} (G_1\, ds_1 + G_2\, ds_2) = \iint_R \left(\frac{\partial G_2}{\partial s_1} - \frac{\partial G_1}{\partial s_2} \right) ds_1\, ds_2,$$

On the other hand, since

$$\iint_S \operatorname{curl} F \cdot d\mathbf{S} = \iint_R \operatorname{curl} F \cdot \frac{\partial r}{\partial s_1} \times \frac{\partial r}{\partial s_2}\, ds_1\, ds_2,$$

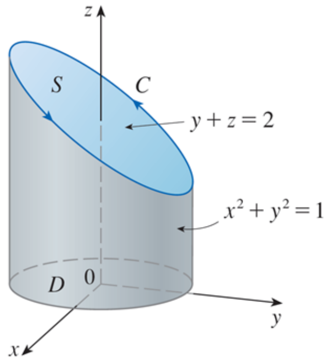
and

$$\begin{aligned}
\operatorname{curl} F \cdot \frac{\partial r}{\partial s_1} \times \frac{\partial r}{\partial s_2} &= \begin{vmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} & \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} & \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \\ \frac{\partial x_1}{\partial s_1} & \frac{\partial x_2}{\partial s_1} & \frac{\partial x_3}{\partial s_1} \\ \frac{\partial x_1}{\partial s_2} & \frac{\partial x_2}{\partial s_2} & \frac{\partial x_3}{\partial s_2} \end{vmatrix} \\
&= \sum_{i,j=1}^{3} \left(\frac{\partial F_j}{\partial x_i} - \frac{\partial F_i}{\partial x_j} \right) \frac{\partial x_i}{\partial s_1} \frac{\partial x_j}{\partial s_2} \quad \text{by the definition of determinant} \\
&= \sum_{i,j=1}^{3} \frac{\partial F_j}{\partial x_i} \frac{\partial x_i}{\partial s_1} \frac{\partial x_j}{\partial s_2} - \sum_{i,j=1}^{3} \frac{\partial F_i}{\partial x_j} \frac{\partial x_j}{\partial s_2} \frac{\partial x_i}{\partial s_1} \\
&= \frac{\partial F}{\partial s_1} \cdot \frac{\partial r}{\partial s_2} - \frac{\partial F}{\partial s_2} \cdot \frac{\partial r}{\partial s_1} \quad \text{by the Chain Rule} \\
&= \frac{\partial G_2}{\partial s_1} - \frac{\partial G_1}{\partial s_2}
\end{aligned}$$

we have

$$\iint_S \operatorname{curl} F \cdot d\mathbf{S} = \iint_R \left(\frac{\partial G_2}{\partial s_1} - \frac{\partial G_1}{\partial s_2} \right) ds_1\, ds_2 = \int_C F \cdot dr$$

Example 1. Evaluate $\int_C F \cdot dr$, where $F(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)



Solution: $\operatorname{curl} F = (1+2y)\,\mathbf{k}$, $S = \{z = g(x,y) = 2-y\}$, $\operatorname{curl} F \cdot dS = (0,0,1+2y)\cdot(-g_x,-g_y,1)\,dA = (1+2y)\,dA$ and $\int_C F\cdot dr = \iint_S \operatorname{curl} F \cdot dS = \iint_D (1+2y)\,dA = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\,r\,dr\,d\theta$.

Example 2. Find the flux of the vector field $F(x,y,z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere

$$x^2 + y^2 + z^2 = 1.$$

Solution: $r(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, $(\phi,\theta) \in D = \{0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$. Then $\iint_S F\cdot d\mathbf{S} = \iint_D F\cdot r_\phi \times r_\theta\,dA = 4\pi/3$.